DETERMINING INCLUSIONS FOR THE MAXWELL'S EQUATIONS

1. Enclosing obstacle

1.1. **Direct Problems and CGO-solutions.** Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{2,1}$-boundary and a connected complement $\mathbb{R}^3 \setminus \bar{\Omega}$. Assume $D \subset \Omega$ is the obstacle or cavity. The electric permittivity ε_0, conductivity σ_0 and magnetic permeability μ_0 have the following properties: there are positive constants $\varepsilon_m, \varepsilon_M, \mu_m, \mu_M$ and σ_M such that for all $x \in \Omega$

$$\varepsilon_m \leq \varepsilon_0(x) \leq \varepsilon_M, \;\; \mu_m \leq \mu_0(x) \leq \varepsilon_M, \;\; 0 \leq \sigma_0(x) \leq \sigma_M$$

and $\varepsilon_0 - \varepsilon_c, \sigma_0, \mu_0 - \mu_c \in C_0^3(\Omega)$ for positive constants ε_c and μ_c.

Considering the boundary value problem of Maxwell's equations

$$\begin{aligned} &\nabla \wedge \mathbf{E} = i\omega\mu_0\mathbf{H}, \;\; \nabla \wedge \mathbf{H} = -i\omega\gamma_0\mathbf{E} \quad \text{in } \; \Omega \setminus \bar{D}, \\ &\nu \wedge \mathbf{E} = f \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega) \quad \text{on } \; \partial\Omega, \end{aligned} \tag{1.1}$$

where $\gamma_0 = \varepsilon_0 + i\frac{\sigma_0}{\omega}$, and zero tangential magnetic field condition on ∂D

$$(\nu \wedge \mathbf{H})|_{\partial D} = 0, \tag{1.2}$$

here ν is the unit outer normal vector on ∂D. Through this note, we assume the non-dissipative case $\sigma_0 = 0$. Then (1.1) degenerates to

$$\nabla \wedge (\mu_0^{-1} \nabla \wedge \mathbf{E}) = \omega^2 \varepsilon_0 \mathbf{E} \quad \text{in } \Omega \setminus \bar{D}. \tag{1.3}$$

Notations. If F is a function space on $\partial\Omega$, the subspace of all those $f \in F^3$ which are tangent to $\partial\Omega$ (orthogonal to the exterior unit normal vector field of $\partial\Omega$) is denoted by TF. For example, for $u \in (H^s(\partial\Omega))^3$ $(s \leq 2)$, we have the decomposition $u = u_t + u_\nu \nu$, where the tangential component $u_t = -\nu \wedge (\nu \wedge u) \in TH^s(\partial\Omega)$ and the normal component $u_\nu = u \cdot \nu \in H^s(\partial\Omega)$. Therefore, we have a decomposition of space $H^s(\partial\Omega)^3 = TH^s(\partial\Omega) \oplus H^s(\partial\Omega)$. For a bounded domain Ω in $\mathbb{R}^3$, we denote

$$TH^{1/2}_{\mathrm{Div}}(\partial\Omega) = \{u \in H^{1/2}(\partial\Omega)^3 \mid \mathrm{Div}(u) \in H^{1/2}(\partial\Omega)\},$$

$$H^1_{\mathrm{Div}}(\Omega) = \{u \in H^1(\Omega)^3 \mid \mathrm{Div}(\nu \wedge u|_{\partial\Omega}) \in H^{1/2}(\partial\Omega)\},$$

with norms

$$\|u\|^2_{TH^{1/2}_{\mathrm{Div}}(\partial\Omega)} = \|u\|^2_{H^{1/2}(\partial\Omega)^3} + \|\mathrm{Div}(u)\|^2_{H^{1/2}(\partial\Omega)},$$

$$\|u\|^2_{H^1_{\text{Div}}(\Omega)} = \|u\|^2_{H^1(\Omega)^3} + \|\text{Div}(\nu \wedge u|_{\partial\Omega})\|^2_{H^{1/2}(\partial\Omega)},$$

where Div is the surface divergence. There are natural inner products making them Hilbert spaces (see [9]). We also have the Hilbert space

$$H(\nabla\wedge, \Omega) = \{u \in L^2(\Omega)^3 \mid \nabla \wedge u \in L^2(\Omega)^3\}$$

with norm

$$\|u\|^2_{H(\nabla\wedge,\Omega)} = \|u\|^2_{L^2(\Omega)^3} + \|\nabla \wedge u\|^2_{L^2(\Omega)^3}.$$

In addition, we define the weighted L^2 space in $\mathbb{R}^3$:

$$L^2_\delta = \left\{ f \in L^2_{loc}(\mathbb{R}^3) \ : \ \|f\|^2_{L^2_\delta} = \int (1+|x|^2)^\delta |f(x)|^2 dx < \infty \right\}.$$

Admissibility. It can be shown that for $f \in TH^{1/2}(\partial\Omega)$ and $g \in TH^{-1/2}(\partial D)$, the boundary value problem of Maxwell's equations

$$\begin{cases} \nabla \wedge \mathbf{E} = i\omega\mu_0 \mathbf{H}, \ \ \nabla \wedge \mathbf{H} = -i\omega\gamma_0 \mathbf{E} \ \ \text{in} \ \ \Omega \setminus \bar{D}, \\ \nu \wedge \mathbf{E}|_{\partial\Omega} = f \\ \nu \wedge \mathbf{H}|_{\partial D} = g, \end{cases} \tag{1.4}$$

has a unique solution $(\mathbf{E}, \mathbf{H}) \in H(\nabla\wedge, \Omega\setminus\bar{D}) \times H(\nabla\wedge, \Omega\setminus\bar{D})$, except for a discrete set of magnetic resonance frequencies $\{\omega_n\}$. A proof for Dirichlet problem can be found in [9]. Moreover, we have the continuous dependency of the $H(\nabla\wedge)$-norm of the solution on the boundary condition

$$\begin{aligned} \|\mathbf{E}\|_{H(\nabla\wedge,\Omega\setminus\bar{D})} &\le C(\|f\|_{H^{1/2}(\partial\Omega)^3} + \|g\|_{H^{-1/2}(\partial D)^3}), \\ \|\mathbf{H}\|_{H(\nabla\wedge,\Omega\setminus\bar{D})} &\le C(\|f\|_{H^{1/2}(\partial\Omega)^3} + \|g\|_{H^{-1/2}(\partial D)^3}). \end{aligned} \tag{1.5}$$

At the same time, a similar proof as in [6] shows that the BVP (1.4) is well posed for $f \in TH^{1/2}_{\text{Div}}(\partial\Omega))$, and $g \in TH^{1/2}_{\text{Div}}(\partial D))$, i.e., except for the resonance frequencies there exists a unique solution $(\mathbf{E}, \mathbf{H}) \in H^1_{\text{Div}}(\Omega \setminus \bar{D}) \times H^1_{\text{Div}}(\Omega \setminus \bar{D})$ s.t.,

$$\|\mathbf{E}\|_{H^1_{\text{Div}}(\Omega\setminus\bar{D})} \le C(\|f\|_{TH^{1/2}_{\text{Div}}(\partial\Omega)} + \|g\|_{TH^{1/2}_{\text{Div}}(\partial D)}),$$
$$\|\mathbf{H}\|_{H^1_{\text{Div}}(\Omega\setminus\bar{D})} \le C(\|f\|_{TH^{1/2}_{\text{Div}}(\partial\Omega)} + \|g\|_{TH^{1/2}_{\text{Div}}(\partial D)}))$$

Let $(\mathbf{E}_0, \mathbf{H}_0)$ denotes the solution without the obstacle.
With well-posedness of the direct problem, the impedance map

$$\Lambda_D(\nu \wedge \mathbf{E}|_{\partial\Omega}) = \nu \wedge \mathbf{H}|_{\partial\Omega},$$

where ν is the unit outer normal on $\partial\Omega$, is bounded from $TH^{1/2}(\partial\Omega)$ to $TH^{-1/2}(\partial\Omega)$ ([9]). Moreover, it is an isomorphism from $TH^{1/2}_{\text{Div}}(\partial\Omega)$ to $TH^{1/2}_{\text{Div}}(\partial\Omega)$, see [6].
The reconstruction of the obstacle will use the CGO-solution constructed in [5].
CGO-solution In [5], the Maxwell's equation was reduced to an 8×8 second

order Schrödinger vector equation by introducing the generalized Sommerfeld potential. A vector CGO-solution (Sommerfeld potential) was constructed for the Schrödinger equation, and the proof of the uniqueness is facilitated compared to [6]. Same technique also appears in dealing with the inverse boundary value problems for Maxwell's equations with partial data [2]. The construction is in $\mathbb{R}^3$.

Define the scalar fields Φ and Ψ as

$$\Phi = \frac{i}{\omega}\nabla\cdot(\gamma_0\mathbf{E}), \quad \Psi = \frac{i}{\omega}\nabla\cdot(\mu_0\mathbf{H}).$$

Under some assumptions on Phi and Psi, we have Maxwell's equation is equivalent to

$$\nabla\wedge\mathbf{E} - \frac{1}{\gamma}\nabla\left(\frac{1}{\mu}\Psi\right) - i\omega\mu\mathbf{H} = 0, \;\; \nabla\wedge\mathbf{H} + \frac{1}{\mu}\nabla\left(\frac{1}{\gamma}\Phi\right) + i\omega\gamma\mathbf{E} = 0.$$

Moreover, in this case, Φ and Ψ vanish. Let $X = (\phi, e, h, \psi)^T \in (\mathcal{D}')^8$ with

$$e = \gamma_0^{1/2}\mathbf{E}, \quad h = \mu_0^{1/2}\mathbf{H},$$

$$\phi = \frac{1}{\gamma_0\mu_0^{1/2}}\Phi, \quad \psi = \frac{1}{\gamma_0^{1/2}\mu_0}\Psi.$$

Then X satisfies

$$(P(i\nabla) - k + V)X = 0, \quad \text{in } \Omega \tag{1.6}$$

where

$$P(i\nabla) = \begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\wedge & 0 \\ 0 & -\nabla\wedge & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix},$$

$$V = (k-\kappa)\mathbf{1}_8 + \left(\begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & -\nabla\wedge & 0 \\ 0 & \nabla\wedge & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix} D\right) D^{-1},$$

$$D = \operatorname{diag}(\mu_0^{1/2}, \gamma_0^{1/2}\mathbf{1}_3, \mu_0^{1/2}\mathbf{1}_3, \gamma_0^{1/2}), \;\; \kappa = \omega(\gamma_0\mu_0)^{1/2}, \;\; k = \omega(\varepsilon_c\mu_c)^{1/2}.$$

A desirable property of this operator is

$$(P(i\nabla) - k + V)(P(i\nabla) + k - V^T) = -(\Delta + k^2)\mathbf{1}_8 + Q,$$

where

$$Q = VP(i\nabla) - P(i\nabla)V^T + k(V + V^T) - VV^T$$

is a zeroth-order matrix multiplier. Based on this, by writing an ansatz for X, we define the generalized Sommerfeld potential Y

$$X = (P(i\nabla) + k - V^T)Y.$$

So it satisfies the Schrödinger equation

$$(-\Delta - k^2 + Q)Y = 0. \tag{1.7}$$

The following CGO-solution is due to the Faddeev's Kernel. Let $\zeta \in \mathbb{C}^3$ be a vector with $\zeta \cdot \zeta = k^2$. Suppose $y_{0,\zeta} \in \mathbb{C}^8$ is a constant vector with respect to x and bounded with respect to ζ, there exist a unique solution of (1.7) of the form

$$Y_\zeta(x) = e^{ix\cdot\zeta}(y_{0,\zeta} - v_\zeta(x)),$$

where $v_\zeta(x) \in (L^2_{\delta+1})^8$, and

$$\|v_\zeta\|_{(L^2_{\delta+1})^8} \leq C/|\zeta|$$

for $\delta \in (-1, 0)$. Moreover, one can show that $v_\zeta \in H^s(\Omega)^8$ for $0 \leq s \leq 2$, e.g., see [1], and

$$\|v_\zeta(x)\|_{H^s(\Omega)^8} \leq C|\zeta|^{s-1}. \tag{1.8}$$

Lemma 3.1 in [5] states that if we choose $y_{0,\zeta}$ such that the first and the last components of $(P(\zeta) - k)y_{0,\zeta}$ vanish, then for large $|\zeta|$, X_ζ provides the solution of the original Maxwell's equation.

Let's examine this X_ζ more closely by giving specific choices of vectors.

$$X_\zeta = e^{ix\cdot\zeta}\left[(P(-\zeta) + k)\, y_{0,\zeta} + \left(P(-\zeta)v_\zeta + P(i\nabla)v_\zeta - V^T y_{0,\zeta} + kv_\zeta - V^T v_\zeta\right)\right]. \tag{1.9}$$

As in [5], we choose

$$y_{0,\zeta} = \frac{1}{|\zeta|}(\zeta \cdot a, ka, kb, \zeta \cdot b)^T,$$

where

$$\zeta = -i\tau\rho + \sqrt{\tau^2 + k^2}\rho^\perp,$$

with $\rho, \rho^\perp \in \mathbb{S}^2$ and $\rho\cdot\rho^\perp = 0$. $\tau > 0$ is used to control the size of $|\zeta| = \sqrt{2\tau^2 + k^2}$. Taking $\tau \to \infty$, we have

$$\frac{\zeta}{|\zeta|} \to \hat{\zeta} = \frac{1}{\sqrt{2}}(-i\rho + \rho^\perp).$$

Choosing a and b such that

$$\hat{\zeta} \cdot b = 1, \quad \hat{\zeta} \cdot a = 0,$$

e.g., when $n \geq 3$, let

$$a \perp \rho, a \perp \rho^\perp; \quad b = \frac{\bar{\hat{\zeta}}}{|\hat{\zeta}|^2}.$$

The choice of $y_{0,\zeta}$ is such that

$$x_{0,\zeta} := (P(-\zeta) + k)y_{0,\zeta} = \frac{1}{|\zeta|}\begin{pmatrix} 0 \\ -(\zeta \cdot a)\zeta - k\zeta \wedge b + k^2 a \\ k\zeta \wedge a - (\zeta \cdot b)\zeta + k^2 b \\ 0 \end{pmatrix}.$$

It's easy to see that

$$\eta = (x_{0,\zeta})_2 \to -k\hat{\zeta} \wedge b \ \ (\sim \mathcal{O}(1)),$$
$$\theta = (x_{0,\zeta})_3 \sim \mathcal{O}(\tau)$$

as $\tau \to \infty$. Then X_ζ is written in the form

$$X_\zeta = e^{\tau(x\cdot\rho)+i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}(x_{0,\zeta} + r_\zeta(x))$$

where

$$r_\zeta = P(-\zeta)v_\zeta + P(i\nabla)v_\zeta - V^T y_{0,\zeta} + kv_\zeta - V^T v_\zeta$$

satisfying for $C > 0$ independent of ζ.

$$\|r_\zeta\|_{L^2_\delta(\Omega)^3} \leq C.$$

Hence the CGO solution of the Maxwell's equation is given by

$$\begin{aligned}\mathbf{E}_0 &= \varepsilon_0^{-1/2} e^{\tau(x\cdot\rho)+i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}(\eta + R)\\ \mathbf{H}_0 &= \mu_0^{-1/2} e^{\tau(x\cdot\rho)+i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}(\theta + Q)\end{aligned}$$

where $\eta = \mathcal{O}(1)$, $\theta = \mathcal{O}(\tau)$, $R, Q \in L^2_\delta(\mathbb{R}^3)$ are bounded for $\tau \gg 1$.

1.2. **Main result.** Adding a parameter t in the weight, we use the CGO solution

$$\begin{aligned}\mathbf{E}_0 &= \varepsilon_0^{-1/2} e^{\tau(x\cdot\rho-t)+i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}(\eta + R)\\ \mathbf{H}_0 &= \mu_0^{-1/2} e^{\tau(x\cdot\rho-t)+i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}(\theta + Q)\end{aligned} \tag{1.10}$$

to define an indicator function and a support function

Definition 1. Define

$$I_\rho(\tau, t) := \int_{\partial\Omega} (\nu \wedge \mathbf{E}_0) \cdot \left(\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \wedge \mathbf{E}_0) \wedge \nu}\right) dS$$

to be the indicator function

Definition 2. Define the support function of the convex hull of D

$$h_D(\rho) := \sup_{x\in D} x \cdot \rho$$

for a fixed $\rho \in S^2$.

Now we are ready to state our main result.

Theorem 1.1. *We assume that the set $\{x \in \mathbb{R}^3 \mid x \cdot \rho = h_D(\rho)\} \cap \partial D$ consists of one point and the Gaussian curvature of ∂D is not vanishing at that point. Then, we can recover $h_D(\rho)$ by*

$$h_D(\rho) = \inf\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} I_\rho(\tau, t) = 0\}.$$

Moreover, if D is strictly convex, then we can reconstruct D.

Remark 1. The proof of Theorem 1.1 mainly consists of showing the following limits:

$$\lim_{\tau\to\infty} I_\rho(\tau,t)=0, \quad \text{when } \ t>h_D(\rho); \tag{1.11}$$

$$\liminf_{\tau\to\infty} I_\rho(\tau,h_D(\rho))=C>0; \tag{1.12}$$

Remark 2. A surface is said to be strictly convex if its Gaussian curvature is everywhere positive. Therefore, if the obstacle D is strictly convex, then Theorem 1.1 provides a reconstruction scheme of the shape of D.

1.3. **A key integral equality.**

Lemma 1.2. *Assume* $(\mathbf{E},\mathbf{H})$ *is a solution of* (1.1) *or* (1.3) *satisfying the boundary condition*

$$\nu\wedge\mathbf{H}|_{\partial D}=0 \quad \textit{and} \quad \nu\wedge\mathbf{E}|_{\partial\Omega}=\nu\wedge\mathbf{E}_0|_{\partial\Omega}.$$

We have

$$\begin{aligned} & i\omega\int_{\partial\Omega}(\nu\wedge\mathbf{E}_0)\cdot\left[\overline{(\nu\wedge\mathbf{H}-\nu\wedge\mathbf{H}_0)}\wedge\nu\right]dS \\ & =\int_{\Omega\setminus\bar D}\mu_0^{-1}|\nabla\wedge\mathbf{E}-\nabla\wedge\mathbf{E}_0|^2-\omega^2\varepsilon_0|\mathbf{E}-\mathbf{E}_0|^2dx \\ & +\int_D\mu_0^{-1}|\nabla\wedge\mathbf{E}_0|^2-\omega^2\varepsilon_0|\mathbf{E}_0|^2dx. \end{aligned} \tag{1.13}$$

Proof: Denote

$$I:=i\omega\int_{\partial\Omega}(\nu\wedge\mathbf{E}_0)\cdot[\overline{(\nu\wedge\mathbf{H}-\nu\wedge\mathbf{H}_0)\wedge\nu}]dS=i\omega\int_{\partial\Omega}(\nu\wedge\mathbf{E}_0)\cdot(\overline{\mathbf{H}-\mathbf{H}_0})dS.$$

First by integration by parts, we have

$$\begin{aligned} & \int_{\Omega\setminus\bar D}\mu_0^{-1}(\nabla\wedge\mathbf{E})\cdot(\overline{\nabla\wedge\mathbf{E}-\nabla\wedge\mathbf{E}_0})-\omega^2\varepsilon_0\mathbf{E}\cdot(\overline{\mathbf{E}-\mathbf{E}_0})dx \\ = & -\left(\int_{\partial\Omega}-\int_{\partial D}\right)(\nu\wedge\mu_0^{-1}(\nabla\wedge\mathbf{E}))\cdot(\overline{\mathbf{E}-\mathbf{E}_0})dS=0 \end{aligned}$$

by the boundary conditions. Adding this to the following equality

$$\begin{aligned} I &= \int_{\partial\Omega}(\nu\wedge\mathbf{E}_0)\cdot(\overline{-i\omega\mathbf{H}+i\omega\mathbf{H}_0})dS \\ &= \int_{\Omega\setminus\bar D}-\mu_0^{-1}(\nabla\wedge\mathbf{E}_0)\cdot(\overline{\nabla\wedge\mathbf{E}})+\omega^2\varepsilon_0\mathbf{E}_0\cdot\overline{\mathbf{E}}dx \\ &\quad +\int_\Omega\mu_0^{-1}|\nabla\wedge\mathbf{E}_0|^2-\omega^2\varepsilon_0|\mathbf{E}_0|^2dx+\int_{\partial D}(\nu\wedge\mathbf{E}_0)\cdot(\overline{-i\omega\mathbf{H}})dS \end{aligned}$$

with the last term vanishing due to the zero-boundary condition on the interface,

$$\int_{\partial D}(\nu\wedge\mathbf{E}_0)\cdot(\overline{-i\omega\mathbf{H}})dS=\int_{\partial D}(\nu\wedge\mathbf{E}_0)\cdot(\overline{-i\omega(\nu\wedge\mathbf{H})\wedge\nu})dS=0,$$

we obtain (1.13). ■

Remark 3. Providing zero tangential electric field instead of magnetic field on the interface

$$(\nu \wedge \mathbf{E})|_{\partial D} = 0,$$

we have a similar identity

$$-I = \int_{\Omega\setminus\bar{D}} \mu_0^{-1}|\nabla\wedge\mathbf{E}-\nabla\wedge\mathbf{E}_0|^2-\omega^2\varepsilon_0|\mathbf{E}-\mathbf{E}_0|^2dx+\int_D \mu_0^{-1}|\nabla\wedge\mathbf{E}_0|^2-\omega^2\varepsilon_0|\mathbf{E}_0|^2dx. \tag{1.14}$$

1.4. **Proof of the main theorem.** First, we show (1.11) by propose an upper bound of the indicator function. Let $\tilde{\mathbf{E}} = \mathbf{E} - \mathbf{E}_0$ be the reflect solution in $\Omega \setminus \bar{D}$. It satisfies

$$\begin{cases} \nabla \wedge (\mu_0^{-1}\nabla \wedge \tilde{\mathbf{E}}) - \omega^2\varepsilon_0\tilde{\mathbf{E}} = 0 \quad \text{in} \quad \Omega \setminus \bar{D}, \\ \nu \wedge \tilde{\mathbf{E}}|_{\partial\Omega} = 0, \\ \nu \wedge (\mu_0^{-1}\nabla \wedge \tilde{\mathbf{E}})|_{\partial D} = -\nu \wedge \mathbf{H}_0|_{\partial D} \in TH^{-1/2}(\partial D). \end{cases} \tag{1.15}$$

The well-posedness of this boundary value problem shows

$$\|\tilde{\mathbf{E}}\|_{H(\nabla\wedge,\Omega\setminus\bar{D})} \le C\|\nu \wedge \mathbf{H}_0|_{\partial D}\|_{H^{-1/2}(\partial D)^3} \le C\|\mathbf{E}_0\|_{H(\nabla\wedge,D)},$$

where we denote the general constant $C > 0$. The second inequality is because for $v \in H(\nabla\wedge, D)$,

$$\begin{aligned} \int_{\partial D}(\nu \wedge \mathbf{H}_0) \cdot v dx &= -\int_D \mathbf{H}_0 \cdot \nabla \wedge v + \nabla \wedge \mathbf{H}_0 \cdot v dx \\ &= \int_D \frac{i}{\omega}\mu^{-1}(\nabla \wedge \mathbf{E}_0) \cdot (\nabla \wedge v) + i\omega\varepsilon_0\mathbf{E}_0 \cdot v dx. \end{aligned}$$

Notice that

$$\|\mathbf{H}_0\|_{H(\nabla\wedge,D)} \le C\|\mathbf{E}_0\|_{H(\nabla\wedge,D)} \le C(\|\mathbf{E}_0\|_{L^2(D)^3}+\|\mathbf{H}_0\|_{L^2(D)^3}) \le C\|\mathbf{H}_0\|_{H(\nabla\wedge,D)}.$$

Therefore, (1.13) implies

$$I_\rho(\tau, t) \le C(\|\mathbf{E}_0\|_{L^2(D)^3} + \|\mathbf{H}_0\|_{L^2(D)^3}). \tag{1.16}$$

Plug in the CGO-solution (1.10), we obtain the following estimates:

$$\|\mathbf{E}_0\|^2_{L^2(D)^3} \le Ce^{2\tau(h_D(\rho)-t)}\|\eta + R\|^2_{L^2(D)^3} \sim e^{2\tau(h_D(\rho)-t)} \quad \tau \gg 1,$$

$$\|\mathbf{H}_0\|^2_{L^2(D)^3} \le Ce^{2\tau(h_D(\rho)-t)}\|\theta + Q\|^2_{L^2(D)^3} \sim \tau^2e^{2\tau(h_D(\rho)-t)} \quad \tau \gg 1.$$

Therefore, we obtain

$$I_\rho(\tau, t) \le C\tau e^{2\tau(h_D(\rho)-t)}$$

for τ large enough, proving the first limit (1.11).

To show the second limit (1.12), it suffices to show the following two lemmas.

Lemma 1.3. *If $t = h_D(\rho)$ in CGO-solution (1.10), then*

$$\liminf_{\tau\to\infty} \int_D |\nabla \wedge \mathbf{E}_0|^2 dx = C,$$

with some constant $C > 0$.

Lemma 1.4. *If $t = h_D(\rho)$, then there exists a positive number c such that*

$$\frac{\omega^2 \varepsilon_0 \left(\int_{\Omega\setminus\bar{D}} |\mathbf{E} - \mathbf{E}_0|^2 dx + \int_D |\mathbf{E}_0|^2 dx \right)}{\mu_0^{-1} \int_D |\nabla \wedge \mathbf{E}_0|^2 dx} \le c < 1,$$

for τ large enough.

Proof of Lemma 1.3: This is the same proof as in the conductivity case by noticing the left hand side integral

$$\int_D |\nabla \wedge \mathbf{E}_0|^2 dx \ge C \|\mathbf{H}_0\|^2_{L^2(D)^3} \ge C \int_D \tau^2 e^{2\tau(x\cdot\rho - h_D(\rho))} dx.$$

Proof of Lemma 1.4: The proof here is essentially the same as for the Helmholtz equation. It suffices to show

$$\lim_{\tau\to\infty} \frac{\omega^2 \varepsilon_0 \int_{\Omega\setminus\bar{D}} |\mathbf{E} - \mathbf{E}_0|^2 dx}{\mu_0^{-1} \int_D |\nabla \wedge \mathbf{E}_0|^2 dx} = \lim_{\tau\to\infty} \frac{\varepsilon_0 \int_{\Omega\setminus\bar{D}} |\mathbf{E} - \mathbf{E}_0|^2 dx}{\mu_0 \int_D |\mathbf{H}_0|^2 dx} = 0.$$

So we estimate the numerator. Consider the BVP

$$\begin{cases} \nabla \wedge p = i\omega\mu_0 q, \quad \nabla \wedge q = -i\omega\varepsilon_0 p - \frac{i}{\omega}\bar{\tilde{\mathbf{E}}} \quad \text{in} \ \ \Omega \setminus \bar{D}, \\ p|_{\partial\Omega} = 0, \\ q|_{\partial D} = 0. \end{cases} \tag{1.17}$$

************or $\nu \wedge q|_{\partial D} = 0$? which boundary conditions can guarantee $p \in H^2(\Omega \setminus \bar{D})$?

Assume we propose a proper zero boundary condition for this BVP such that it is well-posed for $\tilde{\mathbf{E}} \in H(\nabla\wedge, \Omega \setminus \bar{D})$, i.e., there exists $p \in H^2(\Omega \setminus \bar{D})^3$, s.t.,

$$\|p\|_{H^2(\Omega\setminus\bar{D})^3} \le C \|\tilde{\mathbf{E}}\|_{L^2(\Omega\setminus\bar{D})^3}.$$

By the Sobolev embedding, we have

$$|p(x) - p(y)| \le C|x - y|^{1/2} \|\tilde{\mathbf{E}}\|_{L^2(\Omega\setminus\bar{D})^3} \quad \text{for} \ \ x, y \in \Omega \setminus \bar{D},$$

$$\sup_{x\in\Omega\setminus\bar{D}} |p(x)| \le C \|\tilde{\mathbf{E}}\|_{L^2(\Omega\setminus\bar{D})^3}.$$

Notice that

$$\nabla \wedge (\mu_0^{-1} \nabla \wedge p) - \omega^2 \varepsilon_0 p = \bar{\tilde{\mathbf{E}}}.$$

Then, integration by parts shows

$$\begin{aligned}
\int_{\Omega\setminus\bar{D}} |\tilde{\mathbf{E}}|^2 dx &= \int_{\Omega\setminus\bar{D}} \tilde{\mathbf{E}}\cdot(\nabla\wedge(\mu_0^{-1}\nabla\wedge p) - \omega^2\varepsilon_0 p)dx \\
&= \int_{\Omega\setminus\bar{D}} \mu_0^{-1}(\nabla\wedge\tilde{\mathbf{E}})\cdot(\nabla\wedge p) - \omega^2\varepsilon_0\tilde{\mathbf{E}}\cdot p dx \\
&\quad + \left(\int_{\partial\Omega} - \int_{\partial D}\right)\tilde{\mathbf{E}}\cdot(\nu\wedge(\mu_0^{-1}\nabla\wedge p))dS \\
&= \int_{\Omega\setminus\bar{D}} \nabla\wedge(\mu_0^{-1}\nabla\wedge\tilde{\mathbf{E}})\cdot p - \omega^2\varepsilon_0\tilde{\mathbf{E}}\cdot p dx \\
&\quad - \left(\int_{\partial\Omega} - \int_{\partial D}\right)\nu\wedge(\mu_0^{-1}\nabla\wedge\tilde{\mathbf{E}})\cdot p dS \\
&= -\int_{\partial D}\nu\wedge(\mu_0^{-1}\nabla\wedge\mathbf{E}_0)\cdot p dS.
\end{aligned}$$

Denote by x_0 the point in $\{x\in\partial D\,|\,x\cdot\rho = h_D(\rho)\}$. We have

$$\begin{aligned}
\|\tilde{\mathbf{E}}\|^2_{L^2(\Omega\setminus\bar{D})^3} &= \int_{\partial D}(p(x_0)-p(x))\cdot\nu\wedge(\mu_0^{-1}\nabla\wedge\mathbf{E}_0)dS - \int_D \omega^2\varepsilon_0 p(x_0)\cdot\mathbf{E}_0 dx \\
&\le C\left\{\int_{\partial D}|x-x_0|^{1/2}|\nu\wedge\mathbf{H}_0|dS + \int_D|\mathbf{E}_0|dx\right\}\|\tilde{\mathbf{E}}\|_{L^2(\Omega\setminus\bar{D})^3} \\
&\le C\left\{\int_{\partial D}\tau|x-x_0|^{1/2}e^{\tau(x\cdot\rho-h_D(\rho))}dS + \int_D e^{\tau(x\cdot\rho-h_D(\rho))}dx\right\}\|\tilde{\mathbf{E}}\|_{L^2(\Omega\setminus\bar{D})^3}
\end{aligned}$$

This yields

$$\int_{\Omega\setminus\bar{D}}|\tilde{\mathbf{E}}|^2 dx \le C\left\{\tau^2\left(\int_{\partial D}|x-x_0|^{1/2}e^{\tau(x\cdot\rho-h_D(\rho))}dS\right)^2 + \left(\int_D e^{\tau(x\cdot\rho-h_D(\rho))}dx\right)^2\right\}.$$

Then follow the step in Helmholtz case to show

$$\lim_{\tau\to\infty}\tau\int_{\partial D}|x-x_0|^{1/2}e^{\tau(x\cdot\rho-h_D(\rho))}dS = 0,$$

where the assumption of the Gaussian curvature is required.

References

[1] A. Bukhgeim and G. Uhlmann, *Recovering a Potential from Partial Cauchy Data*, Commun. PDE. **27** (2002), no. 3&4, 653-668.

[2] *Inverse Boundary Value Problem for Maxwell's Equations with Local Data* (To appear).

[3] M. Ikehata, *How to draw a picture of an unknown inclusion from boundary measurements: Two mathematical inversion algorithms*, J. Inverse Ill-Posed Problems **7** (1999), 255-271.

[4] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*, Teubner, Stuttgart/Willey, New York, 1986.

[5] P. Ola; E. Somersalo, *Electromagnetic Inverse Problems and Generalized Sommerfeld Potentials*, SIAM J. Appl. Math. **56** (1996), no. 4, 1129-1145.

[6] P. Ola; L. Päivärinta; E. Somersalo, *An Inverse Boundary Value Problem in Electrodynamics*, Duke Math. J. **70** (1993), no. 3, 617-653.

[7] G. Nakamura; K. Yoshida, *Identification of a Non-convex Obstacle for Acoustical Scattering*, J. Inv. Ill-Posed Problems **15** (2007), 1-14.

[8] M. Salo, *Semiclassical Pseudodifferetial Calculus and the Reconstruction of a Magnetic Field*, Comm. PDE **31** (2006), no. 11, 1639-1666.

[9] E. Somersalo; D. Issacson; M. Cheney, *A Linearized Inverse Boundary Value Problem for Maxwell's Equations*, J. Compup. Appl. Math. **42** (1992), 123-136.